Analysis of dynamical systems:

Dynamical systems can be described either as ordinary differential equations in possibly multiple scalar variables, or in state-space form that is a first-order differential equation in a possibly multi-dimensional vector-valued state variable.

- Transforming ODE descriptions to state-space form:

We have already done this before. Let's explore it through a few examples.

Ex 1: Suppose the ODE description of a dynamical system is given by

$$\frac{d^3x}{dt^3} + y^2\frac{d^2x}{dt^2} - y^3 + 3 = 0$$

$$\frac{d^2y}{dt^2} - xy + 7 = 0.$$

$$\boxed{\begin{aligned} &\frac{d^3x}{dt^3} + y^2\frac{d^2x}{dt^2} - y^3 + 3 = 0 \\ &\frac{d^2y}{dt^2} - xy + 7 = 0. \end{aligned}}$$

... can be represented as

$$\dddot{x} + y^2\ddot{x} - y^3 + 3 = 0,$$
$$\ddot{y} - xy + 7 = 0.$$

Step 1: Identify the states.

$$X := \begin{pmatrix} x \\ \dot{x} \\ \ddot{x} \\ y \\ \dot{y} \end{pmatrix}$$

Notice how we start from x, y and go up to derivatives of order $\underline{1}$ less than the max. order in the ODEs.

Step 2: Express $\dot{X}$ as a function of X.

$$\dot{X} = \begin{pmatrix} \dot{x} \\ \ddot{x} \\ \dddot{x} \\ \dot{y} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \ddot{x} \\ y^3 - 3 - y^2\ddot{x} \\ \dot{y} \\ xy - 7 \end{pmatrix}$$

Each row is a function of entries in X. You are done!

Remark: In this dynamical system description, we express $\dot{X}$ as purely a function of X. There is no "input" u in the description.

Two quick definitions:

- Order of a differential equation is the order of the highest derivative in that differential equation.

$$\boxed{\begin{aligned} &\frac{d^3x}{dt^3} + y^2\frac{d^2x}{dt^2} - y^3 + 3 = 0 \\ &\frac{d^2y}{dt^2} - xy + 7 = 0. \end{aligned}}$$

Order: 3

Order: 2

- Order of a dynamical system is the dimension of the state variable.

$$\boxed{X := \begin{pmatrix} x \\ \dot{x} \\ \ddot{x} \\ y \\ \dot{y} \end{pmatrix}}$$

Order: 5.

Ex 2: Consider a scalar dynamical system described by

$$\ddot{x} + f\dot{x} + kx = u,$$

where $u \in \mathbb{R}$ is the input. Express this linear dynamical system in the state-space form $\dot{X} = AX + Bu$, i.e., identify the states X, and the relevant matrices A, B.

$$X = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}.$$

$$\dot{X} = \begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ -f\dot{x} - kx + u \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 0 & 1 \\ -k & -f \end{pmatrix}}_{:= A} \underbrace{\begin{pmatrix} x \\ \dot{x} \end{pmatrix}}_{= X} + \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{:= B} u$$

Remark: Linear ODE's always admit a linear dynamical system description in state-space form.

- <u>Finding system trajectories</u>

Next on our agenda, we would like to compute how the system evolves over time, given an initial point and the input. More precisely, consider a dynamical system with the state space description $\dot{x} = F(x, u)$.

Goal: Given u, $x(0)$, find $x(t)$.

This amounts to integrating $\dot{x} = F(x, u)$.

We have two options:

- Analytically solve the differential equation $\dot{x} = F(x, u)$.
  This in general is difficult to do unless F is of a specific form
- Numerically solve the equation approximately.

Let's consider an example where we can both solve the differential equation analytically and then compare its solution with a numerical scheme.

__Example__: Scalar dynamical system $\dot{x} = x - 3$, starting from $x(0) = 1$.

Analytical solution:

$$\frac{dx}{dt} = x - 3.$$

$$\Rightarrow \int_{x(0)}^{x(t)} \frac{dx}{x-3} = \int_0^t dt$$

$$\Rightarrow \ln\left(\frac{x(t) - 3}{x(0) - 3}\right) = t$$

$$\Rightarrow x(t) = 3 + (x(0) - 3)e^t$$

$$= 3 - 2e^t.$$

Remark: This would not be easy if the dynamical system was given by say

$$\dot{x} = \cot(e^{\sin x}) - 3.$$

Hence, we study numerical methods.

Numerical solution: We will study Euler's method to solve $\dot{x} = x - 3$.

$$\frac{d}{dt} x(t) = x(t) - 3.$$

$$\Rightarrow \underbrace{\lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}} = x(t) - 3$$

Approximate this by $\dfrac{x(t+\Delta t) - x(t)}{\Delta t}$ for a positive Δt.

... The smaller the Δt, the better the approximation.

Using this approximation,

$$x(t+\Delta t) \approx x(t) + \Delta t \cdot (x(t) - 3).$$

Given $x(0) = 1$ with a choice of $\Delta t = 0.1$, let's compute $x(N.\Delta t)$ for $N = 1, 2, \ldots$.

$$x(0.1) \approx x(0) + 0.1\,(x(0) - 3)$$

$$= 1.1\, x(0) - 0.3$$

$$= 0.8$$

$$x(0.2) \approx 1.1\, x(0.1) - 0.3 = 0.58$$

$$x(0.3) \approx 1.1\, x(0.2) - 0.3 = 0.34$$

$$x(0.4) \approx 1.1\, x(0.3) - 0.3 = 0.07$$

$$x(0.5) \approx 1.1\, x(0.4) - 0.3 = -0.22$$

$$\vdots$$

Notice that each step is approximate, and hence, error tends to accumulate as you continue the process for larger number of iterations.

Let's compare the analytical and numerical solutions of $\dot{x} = x - 3$, starting from $x(0) = 1$.

t	$x(t)$ from analytical solution $x(t) = 3 - 2e^t$	$x_E(t)$ from Euler's method	Absolute error of Euler's method $\lvert x_E(t) - x(t) \rvert . 100\%$
0	1	1	0
0.1	0.79	0.8	0.01
0.2	0.56	0.58	0.02
0.3	0.30	0.34	0.04
0.4	0.02	0.07	0.05
0.5	-0.30	-0.22	0.08.

absolute error is increasing with time.

Euler's method for higher-order systems.

Let $\dot{X} = F(X, u)$, where $X \in \mathbb{R}^n$, $u \in \mathbb{R}^m$.

Then, $F : \mathbb{R}^{n+m} \to \mathbb{R}^n$.

- Replace $\dot{X}$ by $\dfrac{X(t+\Delta t) - X(t)}{\Delta t}$.
- Then, compute X iteratively using the relation

$$X(t+\Delta t) = X(t) + \Delta t . F\big(X(t), u(t)\big).$$

Let's do an example.

Suppose $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, and $\dot{X} = \begin{pmatrix} x_1^2 - x_2 \\ x_2^2 + 4 \end{pmatrix}$.

Compute $X(1)$ using $\Delta t = 1$, starting from $X(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$$X(1) = X(0) + \begin{pmatrix} x_1(0)^2 - x_2(0) \\ x_2(0)^2 + 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \end{pmatrix}.$$

$$X(2) = X(1) + \begin{pmatrix} x_1(1)^2 - x_2(1) \\ x_2(1)^2 + 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \end{pmatrix} + \begin{pmatrix} -35 \\ 40 \end{pmatrix} = \begin{pmatrix} -36 \\ 46 \end{pmatrix}.$$

Another example:

Consider the dynamical system described by $\ddot{x} + x^3 = 5$. Compute $x(0.2)$, starting from $x(0) = 1$, $\left.\frac{dx}{dt}\right|_{t=0} = 0$ using Euler's method with a step size of 0.1.

- Transform ODE to state space description:

$$X = \begin{pmatrix} x \\ \dot{x} \end{pmatrix} \Rightarrow \dot{X} = \begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ 5-x^3 \end{pmatrix}.$$

For notational convenience call $x_1 = x$, $x_2 = \dot{x}$.

Then, $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, and $\dot{X} = \begin{pmatrix} x_2 \\ 5-x_1^3 \end{pmatrix}$.

$$X(0.1) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} + 0.1 \cdot \begin{pmatrix} x_2(0) \\ 5-x_1(0)^3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0.1 \begin{pmatrix} 0 \\ 5-1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0.4 \end{pmatrix}.$$

$$X(0.2) = \begin{pmatrix} x_1(0.1) \\ x_2(0.1) \end{pmatrix} + 0.1 \begin{pmatrix} x_2(0.1) \\ 5-x_1(0.1)^3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0.4 \end{pmatrix} + 0.1 \begin{pmatrix} 0.4 \\ 5-1^3 \end{pmatrix}$$

$$= \begin{pmatrix} 1.04 \\ 0.8 \end{pmatrix}.$$ ... Read off $x(0.2) = x_1(0.2) = 1.04$.

Equilibrium pts:

X_e is an equilibrium pt. of a dynamical system $\dot{X} = F(x)$ if $F(X_e) = 0$.

If you start the dynamical system at an equilibrium pt. X_e, notice that $\dot{X} = F(X_e) = 0$, meaning the system remains at X_e forever!.

Example: $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\dot{X} = \begin{pmatrix} x_1 - x_2^2 \\ x_1 - 1 \end{pmatrix}$.

Eq. pts satisfy $x_1 - x_2^2 = 0$ and $x_1 - 1 = 0$.

Solutions are given by $x_1 = 1$, $x_2 = \pm 1$.

$\therefore$ There are 2 eq. pts. $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.